



p -Bessel Pairs, Hardy's Identities and Inequalities and Hardy–Sobolev Inequalities with Monomial Weights

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Abstract

In this paper, we first prove a general symmetrization principle for the Hardy type inequality with non-radial weights of the form $A(|x|)|x_1|^{p_1} \dots |x_N|^{p_N}$ (Theorem 1.1). Using this symmetrization principle for Hardy's inequalities, we can establish the improved L^p -Hardy inequalities with such non-radial monomial weights (Theorem 1.2). Second, we introduce the notion of p -Bessel pairs and give applications to L^p -Hardy identities with non-radial monomial weights (Theorem 1.3) and Hardy inequalities (see Theorem 1.4) and their virtual extremals (see Remark 1.2). (See Theorem 1.5 for the special case $p = 2$ where we derive L^2 -Hardy identities and inequalities with monomial weights which have not been studied in the literature). In the special case when $P = (0, \dots, 0)$, they imply the L^p -Hardy identities and Hardy inequalities with remainder terms on any finite balls and the entire space $\mathbb{R}^N$ (Theorem 1.6), while in the special case $P = (0, \dots, 0, \alpha)$, $\alpha > 0$, our results provide the L^p -Hardy identities and Hardy inequalities on half balls and the half spaces (Theorem 1.7). By taking special examples of p -Bessel pairs, we establish some particular Hardy's identities and weighted Sobolev inequalities which are of independent interest. We also establish the optimal Hardy inequalities with monomial weights and explicit forms of extremal

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functions. (See Corollaries 1.1, 1.2, 1.3, 1.4, 1.5.) Our above results sharpened earlier results in the literature even in the case of L^2 Hardy inequalities. Finally, we establish the sharp constants and optimal functions of the L^p -Hardy–Sobolev inequalities with monomial weights (Theorem 1.8).

Keywords p -Bessel pair · Hardy inequality · Hardy–Sobolev inequality · Monomial weight

Mathematics Subject Classification 42B35 · 42B37 · 26D10 · 46E35 · 35A23

1 Introduction

1.1 Hardy Type Inequalities

The Hardy inequalities are of fundamental importance in many areas of partial differential equations, spectral theory and mathematical physics. The Hardy inequality concerned in this paper has the following form: for $N \geq 2$, $p \geq 1$, $p \neq N$, and $u \in C_0^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left| \frac{N-p}{p} \right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx. \quad (1.1)$$

The constant $\left| \frac{N-p}{p} \right|^p$ is sharp and cannot be attained by nontrivial functions. This is the multidimensional version of the inequality proved by Hardy in [41]: for nonnegative functions u , with $p \geq 1$:

$$\left(\frac{p-1}{p} \right)^p \int_0^\infty \left(\frac{1}{x} \int_0^x u(t) dt \right)^p dx \leq \int_0^\infty u(x)^p dx. \quad (1.2)$$

We also mention here the following weighted Hardy inequality in one-dimensional space (see [42]):

$$\left(\frac{|\alpha|}{p} \right)^p \int_0^\infty x^{\alpha-1} \left| \int_0^x u(t) dt \right|^p dx \leq \int_0^\infty x^{p+\alpha-1} u(x)^p dx. \quad (1.3)$$

Here the constant $\left(\frac{|\alpha|}{p} \right)^p$ is optimal and cannot be achieved by nontrivial functions.

There exist several proofs of (1.1) in the literature. One of the most standard methods to prove (1.1) is to use the Schwarz rearrangement. The argument is as follows: by the well-known Pólya–Szegő inequality and Hardy–Littlewood inequality, we have

$$\left| \frac{N-p}{p} \right|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left| \frac{N-p}{p} \right|^p \int_{\mathbb{R}^N} \frac{|u^*|^p}{|x|^p} dx$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^N} |\nabla u^*|^p \, dx.$$

Here u^* is the radial nonincreasing rearrangement of u . As a consequence, it is now enough to assume that u is radial. Therefore we can convert the problem to the one-dimensional case in which we can apply the one-dimensional Hardy inequality (1.3). However, the downside of the above argument is that it doesn't work for general weights even when the weights are radial.

The first aim of this paper is to establish a general symmetrization principle for the Hardy type inequalities with nonradial weights of the form $A(|x|)x^P$.

Theorem 1.1 *Let $p > 1$, $0 < R \leq \infty$, A and B be positive functions on $(0, R)$. Then the following are equivalent:*

- (A) $\int_{B_R^*} A(|x|) |\nabla u|^p x^P \, dx \geq \int_{B_R^*} B(|x|) |u|^p x^P \, dx$ for all $u \in C_0^\infty(B_R^*)$
- (B) $\int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P \, dx \geq \int_{B_R^*} B(|x|) |u|^p x^P \, dx$ for all $u \in C_0^\infty(B_R^*)$
- (C) $\int_{B_R^*} A(|x|) |\nabla u|^p x^P \, dx \geq \int_{B_R^*} B(|x|) |u|^p x^P \, dx$ for all radial $u \in C_0^\infty(B_R^*)$.

Here $x^P = |x_1|^{P_1} \dots |x_N|^{P_N}$, $P_1 \geq 0, \dots, P_N \geq 0$, is the monomial weight, $\mathbb{R}_*^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i > 0 \text{ whenever } P_i > 0\}$, $B_R^* = B_R \cap \mathbb{R}_*^N$ and $\mathcal{R} := \frac{x}{|x|} \cdot \nabla$.

It is worth noting that the functional and geometric inequalities with monomial weights have been studied intensively recently. For instance, motivated by an open question raised by Brezis [8], Cabré and Ros-Oton proved in [11] a version of the Sobolev inequality with monomial weight and used it to study the problem of the regularity of stable solutions to reaction-diffusion problems of double revolution. Then, in [12], the authors also set up the Sobolev, Morrey, Trudinger and isoperimetric inequalities with monomial weight x^P . The best constants of the Trudinger–Moser inequalities with monomial weights were computed explicitly in [45]. In [1], Bakry et al. combined the stereographic projection and the Curvature–Dimension condition to set up the sharp Sobolev inequality with monomial weight. In [46, 56], mass transport approach was used to study the sharp constants and optimizers for the Gagliardo–Nirenberg inequalities and logarithmic Sobolev inequalities with arbitrary norm and with monomial weights. We also mention that in [14], the author provides a proof for the Hardy–Sobolev-type inequalities with monomial weights. However, the best constant and the extremals for the inequalities were not studied there. L^2 -Hardy inequalities and L^2 -Caffarelli–Kohn–Nirenberg inequalities with weights $\frac{x^P}{|x|^\lambda}$ have also been studied recently in [25]. We also refer the interested reader to [13, 18, 23, 28, 47], to name just a few, for several results about the functional and geometric inequalities with homogeneous weights.

It is also worthy to note that $\mathcal{R}$ is the radial derivative. Indeed, in the polar coordinate, $|\mathcal{R}u| = |\partial_r u(r\sigma)|$ while $|\nabla u| = \left(|\partial_r u(r\sigma)|^2 + \frac{|\nabla_{\mathbb{S}^{N-1}} u(r\sigma)|^2}{r^2} \right)^{\frac{1}{2}}$. Actually, the radial derivation plays an important part in the literature. In particular the Hardy type inequalities with radial gradient have been intensively studied recently. See, [37, 38],

to mention just a few. We also note that the Hardy inequalities in the framework of equalities have also been investigated in [16,19,24,49,50], for instance.

Though L^2 -Hardy type inequalities with radial weights and remainder terms have been extensively studied, not much has been done for such inequalities for non-radial weights. In particular, L^p -Hardy inequalities with remainder terms for $p \neq 2$ are widely open in the case of non-radial weights. Using the symmetrization principle for Hardy's inequalities (Theorem 1.1), we can establish the following improved L^p -Hardy inequalities with non-radial monomial weights:

Theorem 1.2 *Let $1 < p < \infty$ and $\varepsilon \in \mathbb{R}$. We have for all $u \in C_0^\infty(\mathbb{R}_*^N \setminus \{0\})$ that*

$$\begin{aligned} \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\nabla u|^p x^P dx &\geq \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\mathcal{R}u|^p x^P dx \\ &\geq \left| \varepsilon + \frac{D-p}{p} \right|^p \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx. \end{aligned} \quad (1.4)$$

Moreover, the constant $\left| \varepsilon + \frac{D-p}{p} \right|^p$ is sharp and is never achieved by nontrivial functions. Here $D = N + P_1 + \dots + P_N$.

The fact that the optimal constants in the Hardy inequalities cannot be achieved by nontrivial functions has been well-investigated. Also, the problem of finding improved versions for Hardy's inequalities has attracted great attention in the literature. In the pioneering works [10,53], certain improvements of the L^2 -Hardy inequality have been established by adding nonnegative terms to the potential $\left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$. More precisely, Maz'ya studied in [53, Sect. 2.1.7, Corollary 3] a refinement of both the Sobolev and the Hardy inequalities and set up the following result:

Theorem (Maz'ya [53]) *Let $N + k > 2$, $2 < q < \frac{2(N+k)}{N+k-2}$, and $\gamma = -1 + (N+k)\left(\frac{1}{2} - \frac{1}{q}\right)$. Then there exists $c > 0$ such that*

$$\int_{\mathbb{R}^{N+k}} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^{N+k}} \frac{|u|^2}{|y|^2} dx \geq c \left(\int_{\mathbb{R}^{N+k}} |u|^q |y|^{\gamma q} dx \right)^{\frac{2}{q}}$$

for all $u \in C_0^\infty(\mathbb{R}^{N+k})$, subject to the condition $u(0, z) = 0$ in the case $N = 1$. Here $x = (y, z) \in \mathbb{R}^N \times \mathbb{R}^k$.

In [10], in order to study the stability of certain singular solutions of nonlinear elliptic equations, Brezis and Vázquez proved that:

Theorem (Brezis–Vázquez [10]) *For any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and every $u \in H_0^1(\Omega)$,*

$$\int_{\Omega} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 dx \quad (1.5)$$

where $z_0 = 2.4048 \dots$ is the first zero of the Bessel function $J_0(z)$ and ω_N is the volume of the unit ball in $\mathbb{R}^N$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball but is not achieved in the Sobolev space $H_0^1(\Omega)$.

The fact that $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal but is still not achieved by nontrivial functions led Brezis and Vázquez to ask whether the term on the LHS of (1.5) is just the first term of an infinite series of remainder terms. Brezis and Vázquez's question and Maz'ya's result have received huge attention in the literature. The interested reader is referred to [3, 6, 9, 15, 17, 20, 26, 27, 36, 57, 59], to name just a few. We also cite the monographs [2, 40, 43, 44, 53, 58], for instance, that are standard references on the subject. It is also worth mentioning that several open questions and conjectures about the Hardy type inequalities, as well as many other topics, have been proposed by Maz'ya in [54]. We should note that there is another approach to establish Hardy inequalities with remainder terms by using Fourier analysis techniques in Euclidean spaces (see [4, 5]). There have also been recent progress of establishing higher order Hardy–Sobolev–Maz'ya inequalities on half spaces by using Fourier analysis techniques on hyperbolic spaces (see [51, 52]).

In [35], Frank and Seiringer used the notion of the ground state representations to set up the following inequality that improves, extends and unifies several results about the Hardy type inequalities: if the weighted p -Laplace equation

$$-\operatorname{div} \left(A(x) |\nabla \varphi(x)|^{p-2} \nabla \varphi(x) \right) = B(x) \varphi(x)^{p-1}$$

has a positive weak solution φ , then

$$\int A(x) |\nabla u|^p dx \geq \int B(x) |u|^p dx.$$

See also the paper [55] where Muckenhougt pair has been used to study the necessary and sufficient conditions for the validity of the Hardy inequality on one-dimensional space.

In [39], in order to study the L^2 -Hardy type inequalities with radial weights, Ghousoub and Moradifam formulated and applied the notion of Bessel pairs in the L^2 sense and proved that if A and B are positive C^1 -functions on $(0, R)$ such that $(r^{N-1}A, r^{N-1}B)$ is a Bessel pair on $(0, R)$, $0 < R \leq \infty$, then

$$\int_{B_R} A(|x|) |\nabla u|^2 dx \geq \int_{B_R} B(|x|) |u|^2 dx \quad (1.6)$$

for all $u \in C_0^\infty(B_R \setminus \{0\})$. Moreover, if (1.6) holds for all $u \in C_0^\infty(B_R \setminus \{0\})$, then $(r^{N-1}A, cr^{N-1}B)$ is a Bessel pair on $(0, R)$ for some $c > 0$. Here we say that a couple of C^1 -functions (A, B) is a Bessel pair on $(0, R)$ if the ordinary differential equation

$$(Ay')' + By = 0$$

has a positive solution on the interval $(0, R)$. See the book [40] for several examples and properties of Bessel pairs.

In [24], the weighted Hardy type inequalities (1.6) have been investigated further. More precisely, the authors in [24] applied the factorizations of differential operators to set up the following Hardy type identities: Let $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{N-1}A, r^{N-1}B)$ is a Bessel pair on $(0, R)$, that is there exists $\varphi > 0$ on $(0, R)$ such that

$$\left(r^{N-1}A\varphi'\right)' + r^{N-1}B\varphi = 0.$$

Then, we have for all $u \in C_0^\infty(B_R \setminus \{0\})$ that

$$\begin{aligned} \int_{B_R} A(|x|) |\nabla u|^2 dx &= \int_{B_R} B(|x|) |u|^2 dx \\ &\quad + \int_{B_R} A(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{u}{\varphi(|x|)} \right) \right|^2 dx \end{aligned}$$

and

$$\int_{B_R} A(|x|) |\mathcal{R}u|^2 dx = \int_{B_R} B(|x|) |u|^2 dx + \int_{B_R} A(|x|) \varphi^2(|x|) \left| \mathcal{R} \left(\frac{u}{\varphi(|x|)} \right) \right|^2 dx.$$

This result has been investigated further in [49,50] where the Hardy identities have been applied to establish the improved Hardy inequalities with Bessel pairs for general distance functions $d(x)$, $|\nabla d| = 1$, have been proved. The distance function can be the distance to a point, or to the boundary of the domain, or to a surface with codimension α . The following results are in the spirit of those by Maz'ya [53], Brezis and Vázquez [10], Brezis and Marcus [9], Barbatis et al. [3] and others.

Theorem (Lam et al. [49,50]) *Let $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. Assume that for some $\alpha \in \mathbb{R}$, $\Delta d(x) - \frac{\alpha-1}{d(x)}$ exists on $\{0 < d(x) < R\}$ in the sense of distributions and $(r^{\alpha-1}A, r^{\alpha-1}B)$ is a Bessel pair on $(0, R)$. Then for $u \in C_0^\infty(\{0 < d(x) < R\})$:*

$$\begin{aligned} &\int_{0 < d(x) < R} A(d(x)) |\nabla u(x)|^2 dx - \int_{0 < d(x) < R} B(d(x)) |u(x)|^2 dx \\ &= \int_{0 < d(x) < R} A(d(x)) \varphi^2(d(x)) \left| \nabla \left(\frac{u(x)}{\varphi(d(x))} \right) \right|^2 dx \\ &\quad - \int_{0 < d(x) < R} A(d(x)) |u(x)|^2 \left[\Delta d(x) - \frac{\alpha-1}{d(x)} \right] \frac{\varphi'(d(x))}{\varphi(d(x))} dx \end{aligned}$$

and

$$\begin{aligned} & \int_{0 < d(x) < R} A(d(x)) |\nabla d(x) \cdot \nabla u(x)|^2 dx - \int_{0 < d(x) < R} B(d(x)) |u(x)|^2 dx \\ &= \int_{0 < d(x) < R} A(d(x)) \varphi^2(d(x)) \left| \nabla d(x) \cdot \nabla \left(\frac{u(x)}{\varphi(d(x))} \right) \right|^2 dx \\ & \quad - \int_{0 < d(x) < R} A(d(x)) |u(x)|^2 \left[\Delta d(x) - \frac{\alpha - 1}{d(x)} \right] \frac{\varphi'(d(x))}{\varphi(d(x))} dx. \end{aligned}$$

Here φ is the positive solution of

$$\left(r^{\alpha-1} A(r) y'(r) \right)' + r^{\alpha-1} B(r) y(r) = 0$$

on the interval $(0, R)$.

As a consequence, we get

Theorem (Lam et al. [49,50]) *Let $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. Assume that for some $\alpha \in \mathbb{R}$, $\varphi'(d(x)) \left(\Delta d(x) - \frac{\alpha-1}{d(x)} \right) \leq 0$ on $\{0 < d(x) < R\}$ in the sense of distributions, and $(r^{\alpha-1} A, r^{\alpha-1} B)$ is a Bessel pair on $(0, R)$. Then for $u \in C_0^\infty(\{0 < d(x) < R\})$:*

$$\begin{aligned} & \int_{0 < d(x) < R} A(d(x)) |\nabla u(x)|^2 dx - \int_{0 < d(x) < R} B(d(x)) |u(x)|^2 dx \\ & \geq \int_{0 < d(x) < R} A(d(x)) \varphi^2(d(x)) \left| \nabla \left(\frac{u(x)}{\varphi(d(x))} \right) \right|^2 dx \end{aligned}$$

and

$$\begin{aligned} & \int_{0 < d(x) < R} A(d(x)) |\nabla d(x) \cdot \nabla u(x)|^2 dx - \int_{0 < d(x) < R} B(d(x)) |u(x)|^2 dx \\ & \geq \int_{0 < d(x) < R} A(d(x)) \varphi^2(d(x)) \left| \nabla d(x) \cdot \nabla \left(\frac{u(x)}{\varphi(d(x))} \right) \right|^2 dx. \end{aligned}$$

Here φ is the positive solution of

$$\left(r^{\alpha-1} A(r) y'(r) \right)' + r^{\alpha-1} B(r) y(r) = 0$$

on the interval $(0, R)$.

We remark that the L^2 Hardy's identities and inequalities with Bessel pairs on Carnot groups have been established in [29]. Such Hardy's identities and inequalities on hyperbolic spaces and related Riemannian manifolds have also been proved in [33] and [32].

It is worthy to note that the L^p Hardy inequalities with remainder terms are significantly more difficult when $p \neq 2$. The second main goal of this paper is to establish such L^p Hardy inequalities with monomial weights through L^p Hardy identities with monomial weights. Such Hardy identities with monomial weights are new even in the case $p = 2$.

More precisely, our next aim is to set up conditions of pairs (A, B) such that the Hardy inequalities of the form

$$\int_{B_R^*} A(|x|) |\nabla u|^p x^P dx \geq \int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P dx \geq \int_{B_R^*} B(|x|) |u|^p x^P dx$$

hold. More precisely, motivated by the Bessel pairs in L^2 sense introduced by Ghousoub and Moradifam in [40], we introduced the following notion of the Bessel pairs in the L^p sense, or simply called p -Bessel pair.

Definition 1.1 (A, B) is a p -Bessel pair on $(0, R)$ if the ODE $\left(A(r) |y'|^{p-2} y'\right)' + B(r) |y|^{p-2} y = 0$ has a positive solution on $(0, R)$.

Also, for $p > 1$, denote

$$\begin{aligned} C_p(x, y) &= |x|^p - |x - y|^p - p|x - y|^{p-2}(x - y) \cdot y \\ &= |x|^p + (p - 1)|x - y|^p - p|x - y|^{p-2}(x - y) \cdot x. \end{aligned}$$

Then we have

Lemma 1.1 $C_p(x, y) \geq 0$ and $C_p(x, y) = 0$ if and only if $y = 0$.

Then, we will prove the following L^p Hardy identities with monomial weights:

Theorem 1.3 Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{D-1}A, r^{D-1}B)$ is a p -Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R^* \setminus \{0\})$:

$$\begin{aligned} &\int_{B_R^*} A(|x|) |\nabla u|^p x^P dx - \int_{B_R^*} B(|x|) |u|^p x^P dx \\ &= \int_{B_R^*} A(|x|) C_p\left(\nabla u, \varphi \nabla\left(\frac{u}{\varphi}\right)\right) x^P dx \end{aligned}$$

and

$$\begin{aligned} &\int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P dx - \int_{B_R^*} B(|x|) |u|^p x^P dx \\ &= \int_{B_R^*} A(|x|) C_p\left(\mathcal{R}u, \varphi \mathcal{R}\left(\frac{u}{\varphi}\right)\right) x^P dx. \end{aligned}$$

Here $\varphi = \varphi_{p,D;A,B;R}$ is a positive solution of

$$\left(r^{D-1}A(r)|y'|^{p-2}y'\right)' + r^{D-1}B(r)|y|^{p-2}y = 0 \quad (1.7)$$

on $(0, R)$.

Obviously, our result implies the following Hardy type inequality:

Theorem 1.4 *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{D-1}A, r^{D-1}B)$ is a p -Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R^* \setminus \{0\})$:*

$$\int_{B_R^*} A(|x|) |\nabla u|^p x^P dx \geq \int_{B_R^*} B(|x|) |u|^p x^P dx. \quad (1.8)$$

Regarding the optimizers of the Hardy type inequalities, we have the following remarks.

Remark 1.1 By Theorem 1.3 and Lemma 1.1, the equality in (1.8) occur when $u(x) = c\varphi(|x|)$ for some $c \in \mathbb{C}$. Therefore, the Hardy type inequality (1.8) has at most one nontrivial optimizer. Here a function U is called an optimizer of (1.8) if

$$\int_{B_R^*} A(|x|) |\nabla U|^p x^P dx = \int_{B_R^*} B(|x|) |U|^p x^P dx < \infty.$$

Indeed, if there exist nontrivial functions U and V such that

$$\int_{B_R^*} A(|x|) |\nabla U|^p x^P dx = \int_{B_R^*} B(|x|) |U|^p x^P dx < \infty$$

and

$$\int_{B_R^*} A(|x|) |\nabla V|^p x^P dx = \int_{B_R^*} B(|x|) |V|^p x^P dx < \infty.$$

Then, by Theorem 1.3, we have that

$$\int_{B_R^*} A(|x|) C_p \left(\nabla U, \varphi \nabla \left(\frac{U}{\varphi} \right) \right) x^P dx = \int_{B_R^*} A(|x|) C_p \left(\nabla V, \varphi \nabla \left(\frac{V}{\varphi} \right) \right) x^P dx = 0$$

for a fixed positive solution φ of (1.7). By Lemma 1.1, we get $U(x) = c_1\varphi(|x|)$ for some $c_1 \in \mathbb{C}$ and $V(x) = c_2\varphi(|x|)$ for some $c_2 \in \mathbb{C}$. Therefore $U(x) = cV(|x|)$ for some $c \in \mathbb{C}$.

Remark 1.2 If $\int_{B_R^*} B(|x|) |\varphi(|x|)|^p x^P dx$ is finite, then $c\varphi(|x|)$ is the only extremal function for the Hardy type inequality (1.8). In this case, φ is the only positive solution

of (1.7). Indeed, if φ_1 is a positive solution of (1.7), then by Theorem 1.3, we have

$$\begin{aligned} & \int_{B_R^*} A(|x|) C_p \left(\nabla \varphi, \varphi_1 \nabla \left(\frac{\varphi}{\varphi_1} \right) \right) x^P dx \\ &= \int_{B_R^*} A(|x|) |\nabla \varphi|^p x^P dx - \int_{B_R^*} B(|x|) |\varphi|^p x^P dx \\ &= 0. \end{aligned}$$

Therefore, by Lemma 1.1, $\varphi = c\varphi_1$ for some $c \in \mathbb{C}$.

If $\int_{B_R^*} B(|x|) |\varphi(|x|)|^p x^P dx$ is infinite, optimizer for the Hardy type inequality (1.8) does not exist. Indeed, if there exists nontrivial function U such that

$$\int_{B_R^*} A(|x|) |\nabla U|^p x^P dx = \int_{B_R^*} B(|x|) |U|^p x^P dx < \infty.$$

Then, by Remark 1.1, $U = c\varphi$ for some $c \in \mathbb{C}$. Therefore,

$$\int_{B_R^*} B(|x|) |U(|x|)|^p x^P dx = |c|^p \int_{B_R^*} B(|x|) |\varphi(|x|)|^p x^P dx$$

is infinite, which is a contradiction to the assumption that U is an optimizer. Nevertheless, we still can consider $\varphi(|x|)$ as a “virtual” optimizer for the Hardy type inequality (1.8). We also note that the Hardy type inequality (1.8) may have more than one “virtual” optimizer.

It is worth mentioning that when $p = 2$, then $C_2(x, y) = |x|^2 - |x - y|^2 - 2(x - y) \cdot y = |y|^2$. Hence, in this case, we get the following L^2 -Hardy identities and inequalities with monomial weights as a consequence of our Theorem 1.3

Theorem 1.5 *Let $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{D-1}A, r^{D-1}B)$ is a Bessel pair on $(0, R)$, that is there exists $\varphi > 0$ on $(0, R)$ such that*

$$\left(r^{D-1} A \varphi' \right)' + r^{D-1} B \varphi = 0.$$

Then, we have for all $u \in C_0^\infty(B_R^ \setminus \{0\})$ that*

$$\begin{aligned} \int_{B_R^*} A(|x|) |\nabla u|^2 x^P dx &= \int_{B_R^*} B(|x|) |u|^2 x^P dx \\ &\quad + \int_{B_R^*} A(|x|) \left| \varphi(|x|) \nabla \left(\frac{u}{\varphi(|x|)} \right) \right|^2 x^P dx, \end{aligned}$$

$$\begin{aligned} \int_{B_R^*} A(|x|) |\mathcal{R}u|^2 x^P dx &= \int_{B_R^*} B(|x|) |u|^2 x^P dx \\ &\quad + \int_{B_R^*} A(|x|) \left| \varphi(|x|) \mathcal{R} \left(\frac{u}{\varphi(|x|)} \right) \right|^2 x^P dx \end{aligned}$$

and

$$\int_{B_R^*} A(|x|) |\nabla u|^2 x^P dx \geq \int_{B_R^*} A(|x|) |\mathcal{R}u|^2 x^P dx \geq \int_{B_R^*} B(|x|) |u|^2 x^P dx.$$

When $P = \overrightarrow{0}$, we obtain the following L^p -Hardy identities on ball B_R in $\mathbb{R}^N$. When $R = \infty$, then this offers the L^p Hardy identities on the entire $\mathbb{R}^N$.

Theorem 1.6 *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{N-1}A, r^{N-1}B)$ is a p -Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R \setminus \{0\})$:*

$$\int_{B_R} A(|x|) |\nabla u|^p dx - \int_{B_R} B(|x|) |u|^p dx = \int_{B_R} A(|x|) C_p \left(\nabla u, \varphi \nabla \left(\frac{u}{\varphi} \right) \right) dx$$

and

$$\int_{B_R} A(|x|) |\mathcal{R}u|^p dx - \int_{B_R} B(|x|) |u|^p dx = \int_{B_R} A(|x|) C_p \left(\mathcal{R}u, \varphi \mathcal{R} \left(\frac{u}{\varphi} \right) \right) dx.$$

Here φ is the positive solution of

$$\left(r^{N-1} A(r) |y'|^{p-2} y' \right)' + r^{N-1} B(r) |y|^{p-2} y = 0$$

on $(0, R)$.

When $P = \alpha \mathbf{e}_N$, $\alpha > 0$, we get the L^p -Hardy identities on half balls in $\mathbb{R}_+^N$ and on half-space $\mathbb{R}_+^N$ when taking $R = \infty$.

Theorem 1.7 *Let $N \geq 1$, $p > 1$, $0 < R \leq \infty$, A and B be positive C^1 -functions on $(0, R)$. If $(r^{N+\alpha-1}A, r^{N+\alpha-1}B)$ is a p -Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R \cap \mathbb{R}_+^N \setminus \{0\})$:*

$$\begin{aligned} \int_{B_R \cap \mathbb{R}_+^N} A(|x|) |\nabla u|^p x_N^\alpha dx - \int_{B_R \cap \mathbb{R}_+^N} B(|x|) |u|^p x_N^\alpha dx \\ = \int_{B_R \cap \mathbb{R}_+^N} A(|x|) C_p \left(\nabla u, \varphi \nabla \left(\frac{u}{\varphi} \right) \right) x_N^\alpha dx \end{aligned}$$

and

$$\begin{aligned} & \int_{B_R \cap \mathbb{R}_+^N} A(|x|) |\mathcal{R}u|^p x_N^\alpha dx - \int_{B_R \cap \mathbb{R}_+^N} B(|x|) |u|^p x_N^\alpha dx \\ &= \int_{B_R \cap \mathbb{R}_+^N} A(|x|) C_p \left(\mathcal{R}u, \varphi \mathcal{R} \left(\frac{u}{\varphi} \right) \right) x_N^\alpha dx. \end{aligned}$$

Here φ is the positive solution of

$$\left(r^{N+\alpha-1} A(r) |y'|^{p-2} y' \right)' + r^{N+\alpha-1} B(r) |y|^{p-2} y = 0$$

on $(0, R)$.

We can check that $\left(r^{D-1} r^{p\varepsilon}, \left| \varepsilon + \frac{D-p}{p} \right|^p r^{D-1} r^{p(\varepsilon-1)} \right)$ is a p -Bessel pair on $(0, \infty)$ with $\varphi(r) = r^{-\frac{D-p}{p}-\varepsilon}$. Indeed, in this case $\varphi'(r) = -\left(\frac{D-p}{p} + \varepsilon\right) r^{-\frac{D-p}{p}-\varepsilon-1}$. Hence

$$\begin{aligned} & \left(r^{D-1} r^{p\varepsilon} |\varphi'(r)|^{p-2} \varphi'(r) \right)' \\ &= - \left| \varepsilon + \frac{D-p}{p} \right|^{p-2} \left(\varepsilon + \frac{D-p}{p} \right) \left(r^{D-1+p\varepsilon-\left(\frac{D-p}{p}+\varepsilon+1\right)(p-1)} \right)' \\ &= - \left| \varepsilon + \frac{D-p}{p} \right|^{p-2} \left(\varepsilon + \frac{D-p}{p} \right) \left(r^{\frac{D-p}{p}+\varepsilon} \right)' \\ &= - \left| \varepsilon + \frac{D-p}{p} \right|^p r^{\frac{D-p}{p}+\varepsilon-1} \\ &= -r^{D-1} \left| \varepsilon + \frac{D-p}{p} \right|^p r^{-\left(\frac{D-p}{p}+\varepsilon\right)(p-1)} r^{p(\varepsilon-1)}. \end{aligned}$$

Therefore we have the following identities that provide a direct understanding to Theorem 1.2:

Corollary 1.1 *Let $1 < p < \infty$ and $\varepsilon \in \mathbb{R}$. We have for all $u \in C_0^\infty(\mathbb{R}_*^N \setminus \{0\})$ that*

$$\begin{aligned} & \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\nabla u|^p x^P dx - \left| \varepsilon + \frac{D-p}{p} \right|^p \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx \\ &= \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} C_p \left(\nabla u, |x|^{-\frac{D-p}{p}-\varepsilon} \nabla \left(u |x|^{\frac{D-p}{p}+\varepsilon} \right) \right) x^P dx \\ &= \int_{\mathbb{R}_*^N} C_p \left(|x|^\varepsilon \nabla u, |x|^{-\frac{D-p}{p}} \nabla \left(u |x|^{\frac{D-p}{p}+\varepsilon} \right) \right) x^P dx \end{aligned}$$

and

$$\int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\mathcal{R}u|^p x^P dx - \left| \varepsilon + \frac{D-p}{p} \right|^p \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx$$

$$\begin{aligned} &= \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} C_p \left(\mathcal{R}u, |x|^{-\frac{D-p}{p}-\varepsilon} \mathcal{R} \left(u |x|^{\frac{D-p}{p}+\varepsilon} \right) \right) x^P dx \\ &= \int_{\mathbb{R}_*^N} C_p \left(|x|^\varepsilon \mathcal{R}u, |x|^{-\frac{D-p}{p}} \mathcal{R} \left(u |x|^{\frac{D-p}{p}+\varepsilon} \right) \right) x^P dx. \end{aligned}$$

As a consequence of Corollary 1.1 and Lemma 1.1, we have that the equalities in Theorem 1.2 occur when $u(x) = c |x|^{-\frac{D-p}{p}-\varepsilon}$ for some $c \in \mathbb{R}$. However, in this case $\int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx$ is infinite. Therefore, by Remark 1.2, the Hardy inequality (1.4) has no extremizer. However, we can still consider $|x|^{-\frac{D-p}{p}-\varepsilon}$ as the “virtual” optimizer of the Hardy inequality (1.4).

We also provide here some other consequences of our Theorem 1.3. By using the 2-Bessel pair $\left(r^{D-1} r^{2-D}, r^{D-1} r^{2-D} \frac{z_0^2}{R^2} \right)$ on $(0, R)$, $R > 0$, we obtain the following Poincaré identities and weighted Sobolev inequalities:

Corollary 1.2 *For $R > 0$ and $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\begin{aligned} &\int_{B_R^*} \frac{|\nabla u|^2}{|x|^{D-2}} x^P dx - \frac{z_0^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^{D-2}} x^P dx \\ &= \int_{B_R^*} \frac{J_0^2\left(\frac{z_0}{R}|x|\right)}{|x|^{D-2}} \left| \nabla \left(\frac{u}{J_0\left(\frac{z_0}{R}|x|\right)} \right) \right|^2 x^P dx, \\ &\int_{B_R^*} \frac{|\mathcal{R}u|^2}{|x|^{D-2}} x^P dx - \frac{z_0^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^{D-2}} x^P dx \\ &= \int_{B_R^*} \frac{J_0^2\left(\frac{z_0}{R}|x|\right)}{|x|^{D-2}} \left| \mathcal{R} \left(\frac{u}{J_0\left(\frac{z_0}{R}|x|\right)} \right) \right|^2 x^P dx \end{aligned}$$

and

$$\int_{B_R^*} \frac{|\nabla u|^2}{|x|^{D-2}} x^P dx \geq \int_{B_R^*} \frac{|\mathcal{R}u|^2}{|x|^{D-2}} x^P dx \geq \frac{z_0^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^{D-2}} x^P dx.$$

Moreover, the optimizer for the weighted Sobolev inequality

$$\frac{z_0^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^{D-2}} x^P dx \leq \int_{B_R^*} \frac{|\nabla u|^2}{|x|^{D-2}} x^P dx \quad (1.9)$$

exists and is of the form $J_0\left(\frac{z_0}{R}|x|\right)$.

More generally, if $R > 0$, $0 \leq \alpha \leq \frac{D-\lambda-2}{2}$, and z_α is the first zero of the Bessel function of the first kind $J_\alpha(z)$, then by using the 2-Bessel pair $\left(r^{D-1} r^{-\lambda}, r^{D-1} r^{-\lambda} \left[\left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \frac{1}{r^2} + \frac{z_\alpha^2}{R^2} \right] \right)$ on $(0, R)$ with $\varphi(r) =$

$r^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}r\right)$, we get the following Brezis–Vázquez type identities and inequalities with monomial weight

Corollary 1.3 *Let $R > 0$ and $0 \leq \alpha \leq \frac{D-\lambda-2}{2}$. For any $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\begin{aligned} \int_{B_R^*} \frac{|\nabla u|^2}{|x|^\lambda} x^P dx &= \left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \int_{B_R^*} \frac{|u|^2}{|x|^{\lambda+2}} x^P dx + \frac{z_\alpha^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^\lambda} x^P dx \\ &\quad + \int_{B_R^*} \left| |x|^{\frac{2-D}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right) \nabla \left(\frac{u}{|x|^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right)} \right) \right|^2 x^P dx, \\ \int_{B_R^*} \frac{|\mathcal{R}u|^2}{|x|^\lambda} x^P dx &= \left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \int_{B_R^*} \frac{|u|^2}{|x|^{\lambda+2}} x^P dx + \frac{z_\alpha^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^\lambda} x^P dx \\ &\quad + \int_{B_R^*} \left| |x|^{\frac{2-D}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right) \mathcal{R} \left(\frac{u}{|x|^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right)} \right) \right|^2 x^P dx, \end{aligned}$$

and

$$\begin{aligned} \int_{B_R^*} \frac{|\nabla u|^2}{|x|^\lambda} x^P dx &\geq \int_{B_R^*} \frac{|\mathcal{R}u|^2}{|x|^\lambda} x^P dx \\ &\geq \left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \int_{B_R^*} \frac{|u|^2}{|x|^{\lambda+2}} x^P dx + \frac{z_\alpha^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^\lambda} x^P dx. \end{aligned}$$

Moreover, when $0 < \alpha \leq \frac{D-\lambda-2}{2}$, the optimizer for the weighted Brezis–Vázquez inequality

$$\int_{B_R^*} \frac{|\nabla u|^2}{|x|^\lambda} x^P dx \geq \left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \int_{B_R^*} \frac{|u|^2}{|x|^{\lambda+2}} x^P dx + \frac{z_\alpha^2}{R^2} \int_{B_R^*} \frac{|u|^2}{|x|^\lambda} x^P dx \quad (1.10)$$

exists and is of the form $|x|^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}|x|\right)$.

$$\text{For } \lambda < D - p, \text{ we have } \left(r^{D-1} \frac{1}{r^\lambda}, r^{D-1} \left(\frac{D-p-\lambda}{p} \right)^p \frac{1}{r^{\lambda+p} \left(1 - \left(\frac{R}{r} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} \right)$$

is a p -Bessel pair on $(0, R)$ with $\varphi(r) = R^{-\frac{D-p-\lambda}{p}} \left(\left(\frac{r}{R} \right)^{-\frac{D-p-\lambda}{p-1}} - 1 \right)^{\frac{p-1}{p}} = \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}}$. Hence, we obtain

Corollary 1.4 *Let $\lambda < D - p$ and $R > 0$. For any $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\int_{B_R^*} \frac{1}{|x|^\lambda} |\nabla u|^p x^P dx - \left(\frac{D-p-\lambda}{p} \right)^p \int_{B_R^*} \frac{1}{|x|^{\lambda+p} \left(1 - \left(\frac{R}{|x|} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} |u|^p x^P dx$$

$$\begin{aligned}
 &= \int_{B_R^*} C_p \left(\nabla u, \left(|x|^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}} \nabla \left(\frac{u}{\left(|x|^{-\frac{D-p-\lambda}{p-1}} - |x|^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}}} \right) \right) x^P dx, \\
 &\int_{B_R^*} \frac{1}{|x|^\lambda} |\mathcal{R}u|^p x^P dx - \left(\frac{D-p-\lambda}{p} \right)^p \int_{B_R^*} \frac{1}{|x|^{\lambda+p} \left(1 - \left(\frac{R}{|x|} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} |u|^p x^P dx \\
 &= \int_{B_R^*} C_p \left(\mathcal{R}u, \left(|x|^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}} \mathcal{R} \left(\frac{u}{\left(|x|^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}}} \right) \right) x^P dx,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{B_R^*} \frac{1}{|x|^\lambda} |\nabla u|^p x^P dx &\geq \int_{B_R^*} \frac{1}{|x|^\lambda} |\mathcal{R}u|^p x^P dx \\
 &\geq \left(\frac{D-p-\lambda}{p} \right)^p \int_{B_R^*} \frac{1}{|x|^{\lambda+p} \left(1 - \left(\frac{R}{|x|} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} |u|^p x^P dx. \quad (1.11)
 \end{aligned}$$

In the limiting case $\lambda = D - p$, we have that $\left(r^{D-1} \frac{1}{r^{D-p}}, r^{D-1} \left(\frac{p-1}{p} \right)^p \frac{1}{r^D |\ln \frac{r}{R}|^p} \right)$ is a p -Bessel pair on $(0, R)$ with $\varphi(r) = \left| \ln \frac{r}{R} \right|^{\frac{p-1}{p}}$. Therefore, as a consequence of our main results, we obtain the following Hardy identities and inequalities with monomial weights:

Corollary 1.5 *For any $R > 0$ and $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\begin{aligned}
 &\int_{B_R^*} \frac{1}{|x|^{D-p}} |\nabla u|^p x^P dx - \left(\frac{p-1}{p} \right)^p \int_{B_R^*} \frac{1}{|x|^D \left| \ln \frac{|x|}{R} \right|^p} |u|^p x^P dx \\
 &= \int_{B_R^*} C_p \left(\nabla u, \left| \ln \frac{|x|}{R} \right|^{\frac{p-1}{p}} \nabla \left(\frac{u}{\left| \ln \frac{|x|}{R} \right|^{\frac{p-1}{p}}} \right) \right) x^P dx, \\
 &\int_{B_R^*} \frac{1}{|x|^{D-p}} |\mathcal{R}u|^p x^P dx - \left(\frac{p-1}{p} \right)^p \int_{B_R^*} \frac{1}{|x|^D \left| \ln \frac{|x|}{R} \right|^p} |u|^p x^P dx \\
 &= \int_{B_R^*} C_p \left(\mathcal{R}u, \left| \ln \frac{|x|}{R} \right|^{\frac{p-1}{p}} \mathcal{R} \left(\frac{u}{\left| \ln \frac{|x|}{R} \right|^{\frac{p-1}{p}}} \right) \right) x^P dx,
 \end{aligned}$$

$$\text{and } \int_{B_R^*} \frac{1}{|x|^{D-p}} |\nabla u|^p x^P dx \geq \int_{B_R^*} \frac{1}{|x|^{D-p}} |\mathcal{R}u|^p x^P dx$$

$$\geq \left(\frac{p-1}{p}\right)^p \int_{B_R^*} \frac{1}{|x|^D \left|\ln \frac{|x|}{R}\right|^p} |u|^p x^P dx. \quad (1.12)$$

In particular, when $p = D = N + P_1 + \cdots + P_N$, then we have that $\left(r^{D-1}, r^{D-1} \left(\frac{D-1}{D}\right)^D \frac{1}{r^D \left|\ln \frac{r}{R}\right|^D}\right)$ is a D -Bessel pair on $(0, R)$ with $\varphi = \left|\ln \frac{r}{R}\right|^{\frac{D-1}{D}}$. As a consequence, we get the following critical Hardy identities and inequalities with monomial weights:

Corollary 1.6 *For any $R > 0$ and $u \in C_0^\infty(B_R^* \setminus \{0\})$, there holds*

$$\begin{aligned} & \int_{B_R^*} |\nabla u|^D x^P dx - \left(\frac{D-1}{D}\right)^D \int_{B_R^*} \frac{1}{|x|^D \left|\ln \frac{|x|}{R}\right|^D} |u|^D x^P dx \\ &= \int_{B_R^*} C_D \left(\nabla u, \left|\ln \frac{r}{R}\right|^{\frac{D-1}{D}} \nabla \left(\frac{u}{\left|\ln \frac{|x|}{R}\right|^{\frac{D-1}{D}}} \right) \right) x^P dx, \\ & \int_{B_R^*} |\mathcal{R}u|^D x^P dx - \left(\frac{D-1}{D}\right)^D \int_{B_R^*} \frac{1}{|x|^D \left|\ln \frac{|x|}{R}\right|^D} |u|^D x^P dx \\ &= \int_{B_R^*} C_D \left(\mathcal{R}u, \left|\ln \frac{r}{R}\right|^{\frac{D-1}{D}} \mathcal{R} \left(\frac{u}{\left|\ln \frac{|x|}{R}\right|^{\frac{D-1}{D}}} \right) \right) x^P dx, \end{aligned}$$

and

$$\int_{B_R^*} |\nabla u|^D x^P dx \geq \int_{B_R^*} |\mathcal{R}u|^D x^P dx \geq \left(\frac{D-1}{D}\right)^D \int_{B_R^*} \frac{1}{|x|^D \left|\ln \frac{|x|}{R}\right|^D} |u|^D x^P dx.$$

We remark at the end of this subsection that L^p Hardy's identities and inequalities with Bessel pairs on bounded domains and Riemannian manifolds have also been established in [30,31].

1.2 Hardy–Sobolev Inequality

Another inequality that has also been studied widely in the literature is the Sobolev inequality. Actually, Sobolev type inequalities are one of the most used inequalities in analysis and play key roles in many areas of mathematics. In this paper, we concern the following type of Sobolev inequality:

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq S_{p,N} \left(\int_{\mathbb{R}^N} |u(x)|^{\frac{Np}{N-p}} dx \right)^{\frac{N-p}{N}}.$$

We note that we can consider Hardy's inequality as a linear inequality in the sense that it involves the p -th power of u on both sides. In the same sense, Sobolev's inequality can be regarded as a nonlinear inequality, where u appears with a higher power on the right than on the left side. Hence, intuitively, we may think that the Sobolev inequality is stronger than the Hardy inequality. Surprisingly, in [34,40], it was showed that by applying the symmetrization, the Sobolev inequality, except the optimal constant, can be derived from the Hardy inequality. Therefore, by Hölder's inequality, we can also derive the following Hardy–Sobolev inequality as an interpolation between Hardy's and Sobolev inequalities:

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq HS_{p,N,s} \left(\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{(N-s)p}{N-p}}}{|x|^s} \, dx \right)^{\frac{N-p}{N-s}}.$$

In the same spirit, we can show that non-optimal Sobolev inequality and Hardy–Sobolev inequality with monomial weights can be derived from our Hardy's inequality with monomial weight. More precisely, using the Hardy inequality with monomial weights (Theorem 1.2), we will prove the following Sobolev and Hardy–Sobolev type inequalities

Corollary 1.7 *For $1 < p < D$ and $0 \leq s < p$, there exist $S_{p,N,p} > 0$ and $HS_{p,s,N,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R}_*^N)$:*

$$S_{p,N,p} \left(\int_{\mathbb{R}_*^N} |u(x)|^{\frac{Dp}{D-p}} x^P \, dx \right)^{\frac{D-p}{D}} \leq \int_{\mathbb{R}_*^N} |\nabla u|^p x^P \, dx \quad (1.13)$$

and

$$HS_{p,s,N,p} \left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^*(s)}}{|x|^s} x^P \, dx \right)^{\frac{p}{p_D^*(s)}} \leq \int_{\mathbb{R}_*^N} |\nabla u|^p x^P \, dx \quad (1.14)$$

where $p_D^*(s) = \frac{D-s}{D-p} p$.

It is worth noting that the sharp constant and extremal functions of the Sobolev inequality with monomial weight (1.13) have been computed explicitly in [12] using a new isoperimetric inequality involving the weight x^P and with optimal constant. Our purpose here is to present a simple proof for (1.13) without the sharp constant using our Hardy inequality with monomial weight. Our proof relies on the following rearrangement result that can be found in [12,45]:

Lemma 1.2 *Let u be a Lipschitz continuous function in $\mathbb{R}_*^N$ such that $\int_{\{x \in \mathbb{R}_*^N : |u(x)| > t\}} x^P \, dx$ is finite for every positive t . Then there exists a radial rearrangement u^* of u such that*

$$(i) \quad \int_{\{x \in \mathbb{R}_*^N : |u| > t\}} x^P \, dx = \int_{\{x \in \mathbb{R}_*^N : u^* > t\}} x^P \, dx \text{ for all } t,$$

- (ii) u^* is radially decreasing
 (iii) for every Young function Φ (that is, Φ maps $[0, \infty)$ into $[0, \infty)$, vanishes at 0, and is convex and increasing):

$$\int_{\mathbb{R}_*^N} \Phi(|\nabla u^*|) x^P dx \leq \int_{\mathbb{R}_*^N} \Phi(|\nabla u|) x^P dx.$$

- (iv) If $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing, then $\int_{\mathbb{R}_*^N} \Psi(u) x^P dx = \int_{\mathbb{R}_*^N} \Psi(u^*) x^P dx$.

We note that the non-optimal Hardy-Sovolev inequality with monomial weights has been investigated in [14]. However, the optimal constant of the Hardy-Sobolev inequality with monomial weight (1.14) is still missing in the literature. Our last goal of this article is to fill this gap. More precisely, let $1 < p < D$, $0 \leq s < p$ and

$$HS_{p,s,N,P} = \inf \left\{ \frac{\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx}{\left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^{(s)}}}{|x|^s} x^P dx \right)^{\frac{D-p}{D-s}}}; u \in W_0^{1,p}(\mathbb{R}_*^N, x^P dx) \right\}.$$

Then we will show that

Theorem 1.8 *We have*

$$HS_{p,s,N,P} = \left(\frac{1}{2^{k-1}} \frac{\Gamma\left(\frac{P_1+1}{2}\right) \Gamma\left(\frac{P_2+1}{2}\right) \cdots \Gamma\left(\frac{P_n+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \right)^{\frac{p-s}{D-s}} \\ \times (D-s)^{\frac{D-p}{D-s}} \left(\frac{D-p}{p-1} \right)^{p-1} \left[\frac{\Gamma\left(\frac{D-s}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)}{\Gamma\left(\frac{(D-s)p}{p-s}\right)} \right]^{\frac{p-s}{D-s}}.$$

Here k is the number of strictly positive entries of P . Also, $HS_{p,s,N,P}$ can be attained by functions of the form $u(x) = c \left(d + |x|^{\frac{p-s}{p-1}} \right)^{-\frac{D-p}{p-s}}$, $c \in \mathbb{R}$, $d > 0$.

We remark that an extension of Theorem 1.8 with an additional power weight in the gradient term in the expression of $HS_{p,s,N,P}$ has been given in [60] by using a quasi-conformal map type change of variables techniques used earlier in [21,22,48].

The paper is organized as follows: In Sect. 2, we will prove the general symmetrization principle for Hardy type inequalities (Theorem 1.1) and use it to establish Theorem 1.2. In Sect. 3, we introduce p -Bessel pairs and set up several Hardy's identities and inequalities with monomial weights. The optimizers and extremal functions of Hardy-Sobolev inequality with monomial weights will be investigated in Sect. 4. Finally, in Sect. 5, we present the proofs of Lemma 1.1 and consequences of our main results, namely Corollaries 1.2, 1.3, 1.4, 1.5 and 1.7.

2 Hardy Inequalities with Monomial Weights—Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Obviously $(A) \implies (C)$, $(B) \implies (A)$ so we just need to prove $(C) \implies (B)$.

Let us denote in the polar coordinate that

$$x^P = r^{|P|} \varphi_P(\sigma)$$

and let

$$U(r) = \left(\frac{1}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma} \int_{\mathbb{S}_*^{N-1}} |u(r\sigma)|^p \varphi_P(\sigma) \, d\sigma \right)^{\frac{1}{p}}.$$

Then by Hölder's inequality, we obtain

$$\begin{aligned} \left| pU^{p-1}(r)U'(r) \right| &= \frac{p}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma} \int_{\mathbb{S}_*^{N-1}} |u|^{p-1} \varphi_P(\sigma)^{\frac{p-1}{p}} \left| \frac{\partial u}{\partial r} \right| \varphi_P(\sigma)^{\frac{1}{p}} \, d\sigma \\ &\leq \frac{p}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma} \left(\int_{\mathbb{S}_*^{N-1}} \left| \frac{\partial u}{\partial r} \right|^p \varphi_P(\sigma) \, d\sigma \right)^{1/p} \\ &\quad \times \left(\int_{\mathbb{S}_*^{N-1}} |u|^p \varphi_P(\sigma) \, d\sigma \right)^{1-\frac{1}{p}} \\ &= \frac{pU^{p-1}}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma} \left(\int_{\mathbb{S}_*^{N-1}} \left| \frac{\partial u}{\partial r} \right|^p \varphi_P(\sigma) \, d\sigma \right)^{1/p} \\ &\quad \times \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{1-1/p} \\ &= \frac{pU^{p-1}}{\left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{1/p}} \\ &\quad \times \left(\int_{\mathbb{S}_*^{N-1}} \left| \frac{\partial u}{\partial r} \right|^p \varphi_P(\sigma) \, d\sigma \right)^{1/p}. \end{aligned}$$

Hence

$$|U'(r)|^p \leq \frac{1}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma} \left(\int_{\mathbb{S}_*^{N-1}} \left| \frac{\partial u}{\partial r} \right|^p \varphi_P(\sigma) \, d\sigma \right).$$

Therefore, for $u \in C_0^\infty(B_R^*)$, we have by the polar coordinate that

$$\begin{aligned} \int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P dx &= \int_0^R A(r) r^{N-1} r^{|P|} \int_{\mathbb{S}_*^{N-1}} \left| \frac{\partial u}{\partial r} \right|^p \varphi_P(\sigma) d\sigma dr \\ &\geq \int_0^R A(r) r^{N-1} r^{|P|} \int_{\mathbb{S}_*^{N-1}} |U'(r)|^p \varphi_P(\sigma) d\sigma dr \\ &= \int_{B_R^*} A(|x|) |\nabla U(x)|^p x^P dx. \end{aligned}$$

Since U is radial, using (C), we get

$$\int_{B_R^*} A(|x|) |\nabla U(x)|^p x^P dx \stackrel{\text{by (C)}}{\geq} \int_{B_R^*} B(|x|) |U(x)|^p x^P dx.$$

On the other hand,

$$\begin{aligned} &\int_{B_R^*} B(|x|) |U(x)|^p x^P dx \\ &= \int_{\mathbb{S}_*^{N-1}} \int_0^R B(r) r^{N-1} |U(r)|^p r^{|P|} \varphi_P(\sigma) d\sigma dr \\ &= \int_{\mathbb{S}_*^{N-1}} \int_0^R B(r) r^{N-1} \left(\frac{1}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma} \int_{\mathbb{S}_*^{N-1}} |u(r\sigma)|^p \varphi_P(\sigma) d\sigma \right) r^{|P|} \varphi_P(\sigma) d\sigma dr \\ &= \frac{1}{\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma} \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \int_0^R B(r) r^{N-1} r^{|P|} \left(\int_{\mathbb{S}_*^{N-1}} |u(r\sigma)|^p |\varphi_P(\sigma)| d\sigma \right) dr \\ &= \int_0^R B(r) r^{N-1} r^{|P|} \left(\int_{\mathbb{S}_*^{N-1}} |u(r\sigma)|^p |\varphi_P(\sigma)| d\sigma \right) dr \\ &= \int_{\mathbb{S}_*^{N-1}} \int_0^R B(r) |u(r\sigma)|^p r^{N-1} r^{|P|} |\varphi_P(\sigma)| dr d\sigma \\ &= \int_{B_R^*} B(|x|) |u(x)|^p x^P dx. \end{aligned}$$

Hence, we can deduce that

$$\int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P dx \geq \int_{B_R^*} B(|x|) |u(x)|^p x^P dx.$$

□

Proof of Theorem 1.2 Using Theorem 1.1, we can assume that u is a radial function. Then, by polar coordinate, we have

$$\int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx = \int_{\mathbb{S}_*^{N-1}} \int_0^\infty r^{p(\varepsilon-1)} |u|^p r^{|P|} \varphi_P(\sigma) r^{N-1} dr d\sigma$$

$$= \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \int_0^\infty r^{p(\varepsilon-1)} |u|^p r^{D-1} dr.$$

Note that by integration by parts, we obtain

$$\begin{aligned} & \int_0^\infty r^{p(\varepsilon-1)+D-1} |u|^p dr \\ &= -\frac{p}{p(\varepsilon-1)+1+D-1} \int_0^\infty |u|^{p-1} \left| \frac{\partial u}{\partial r} \right| r^{p(\varepsilon-1)+1+D-1} dr \\ &\leq \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right| \int_0^\infty |u|^{p-1} \left| \frac{\partial u}{\partial r} \right| r^{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} dr \\ &= \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right| \int_0^\infty |u|^{\frac{p}{q}} \left| \frac{\partial u}{\partial r} \right| r^{p\left(\varepsilon - \frac{1}{q}\right) + \varepsilon - \varepsilon + D - 1} dr \\ &= \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right| \int_0^\infty |u|^{\frac{p}{q}} r^{\frac{p}{q}(\varepsilon-1) + \frac{D-1}{q}} \left(\left| \frac{\partial u}{\partial r} \right| r^{\varepsilon + \frac{D-1}{p}} \right) dr \\ &= \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right| \int_0^\infty \left(|u|^p r^{p(\varepsilon-1)+D-1} \right)^{\frac{1}{q}} \left(\left| \frac{\partial u}{\partial r} \right|^p r^{p\varepsilon+D-1} \right)^{\frac{1}{p}} dr. \end{aligned}$$

Here $q = \frac{p}{p-1}$. By Hölder's inequality

$$\begin{aligned} & \int_0^\infty r^{p(\varepsilon-1)+D-1} |u|^p dr \\ &\leq \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right| \left(\int_0^\infty \left(|u|^p r^{p(\varepsilon-1)+D-1} \right) dr \right)^{\frac{1}{q}} \left(\int_0^\infty \left| \frac{\partial u}{\partial r} \right|^p r^{p\varepsilon+D-1} dr \right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} & \int_0^\infty r^{p(\varepsilon-1)+D-1} |u|^p dr \\ &\leq \left(\frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right)^p \int_0^\infty \left| \frac{\partial u}{\partial r} \right|^p r^{p\varepsilon+D-1} dr. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx \\ &\leq \left| \frac{p}{p\left(\varepsilon - \frac{1}{q}\right) + D - 1} \right|^p \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \int_0^\infty \left| \frac{\partial u}{\partial r} \right|^p r^{p\varepsilon+D-1} dr \end{aligned}$$

$$= \left| \frac{p}{p \left(\varepsilon - \frac{1}{q} \right) + D - 1} \right|^p \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\nabla u|^p x^P dx.$$

Therefore, we get that for all $u \in C_0^\infty(\mathbb{R}_*^N \setminus \{0\})$:

$$\begin{aligned} \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\nabla u|^p x^P dx &\geq \int_{\mathbb{R}_*^N} |x|^{p\varepsilon} |\mathcal{R}u|^p x^P dx \\ &\geq \left| \frac{p \left(\varepsilon - \frac{1}{q} \right) + D - 1}{p} \right|^p \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx \\ &= \left| \varepsilon + \frac{D-p}{p} \right|^p \int_{\mathbb{R}_*^N} |x|^{p(\varepsilon-1)} |u|^p x^P dx. \end{aligned}$$

The constant $\left| \varepsilon + \frac{D-p}{p} \right|^p$ is sharp and unachievable since $\left(\frac{|\alpha|}{p} \right)^p$ is optimal and unattainable in (1.3). $\square$

3 Hardy Inequalities with p -Bessel Pairs—Proofs of Theorem 1.3

Proof of Theorem 1.3 By polar coordinate, one has

$$\int_{B_R^*} B(|x|) |u|^p x^P dx = \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} B(r) |u|^p dr d\sigma.$$

One notes that

$$\begin{aligned} \int_0^R r^{D-1} B(r) |u|^p dr &= \int_0^R r^{D-1} B(r) |\varphi\psi|^p dr \\ &= \int_0^R |\psi|^p \varphi r^{D-1} B(r) |\varphi|^{p-2} \varphi dr \\ &= - \int_0^R |\psi|^p \varphi \left(r^{D-1} A(r) |\varphi'|^{p-2} \varphi' \right)' dr \\ &= \int_0^R r^{D-1} A(r) |\varphi'|^{p-2} \varphi' (|\psi|^p \varphi)' dr \\ &= \int_0^R r^{D-1} A(r) |\varphi'|^p |\psi|^p dr \\ &\quad + p \int_0^R r^{D-1} A(r) |\varphi'|^{p-2} \varphi' \varphi |\psi|^{p-2} \psi \psi' dr. \end{aligned}$$

Hence,

$$\begin{aligned}
 & \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} B(r) |u|^p \, dr \, d\sigma \\
 &= \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} A(r) |\psi\varphi' + \varphi\psi'|^p \, dr \, d\sigma \\
 &+ \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} A(r) |\psi\varphi'|^p \, dr \, d\sigma \\
 &+ p \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} A(r) |\psi\varphi'|^{p-2} \psi\varphi'\varphi\psi' \, dr \\
 &- \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R r^{D-1} A(r) |\psi\varphi' + \varphi\psi'|^p \, dr \, d\sigma \\
 &= \int_{B_R^*} A(|x|) |\mathcal{R}u|^p x^P \, dx - \int_{B_R^*} A(|x|) C_p \left(\mathcal{R}u, \varphi \mathcal{R} \left(\frac{u}{\varphi} \right) \right) x^P \, dx.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \int_{B_R^*} A(|x|) C_p \left(\nabla u, \varphi(|x|) \nabla \left(\frac{u}{\varphi(|x|)} \right) \right) x^P \, dx \\
 &= \int_{B_R^*} A(|x|) |\nabla u|^p x^P \, dx - \int_{B_R^*} A(|x|) |\psi\varphi'|^p x^P \, dx \\
 &- p \int_{B_R^*} A(|x|) |\psi\varphi'|^{p-2} \psi\varphi'\varphi(|x|) \mathcal{R}\psi x^P \, dx
 \end{aligned}$$

In the polar coordinate

$$\begin{aligned}
 & \int_{B_R^*} A(|x|) |\psi\varphi'|^{p-2} \psi\varphi'\varphi \mathcal{R}\psi x^P \, dx \\
 &= \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \int_0^R A(r) |\psi\varphi'|^{p-2} \psi\varphi'\varphi \partial_r \psi r^{D-1} \, dr \, d\sigma
 \end{aligned}$$

By integration by parts

$$\begin{aligned}
 & - \int_0^R A(r) |\psi\varphi'|^{p-2} \psi\varphi'\varphi \partial_r \psi r^{D-1} \, dr \\
 &= \int_0^R \psi \partial_r \left(A(r) |\psi\varphi'|^{p-2} \psi\varphi'\varphi r^{D-1} \right) \, dr \\
 &= \int_0^R \psi \left[\begin{aligned} & (p-1) A(r) |\psi|^{p-2} \psi' \varphi r^{D-1} |\varphi'|^{p-2} \varphi' \\ & + A(r) |\psi|^{p-2} \psi \varphi' r^{D-1} |\varphi'|^{p-2} \varphi' \\ & + |\psi|^{p-2} \psi \varphi \left(A(r) r^{D-1} |\varphi'|^{p-2} \varphi' \right)' \end{aligned} \right] \, dr
 \end{aligned}$$

Hence,

$$\begin{aligned} & - \int_0^R A(r) |\psi \varphi'|^p r^{D-1} dr - p \int_0^R A(r) |\psi \varphi'|^{p-2} \psi \varphi' \varphi \partial_r \psi r^{D-1} dr \\ & = \int_0^R |\psi|^p \varphi \left(A(r) r^{D-1} |\varphi'|^{p-2} \varphi' \right)' \\ & = - \int_0^R |\psi|^p B(r) |\varphi|^p r^{D-1} dr \end{aligned}$$

Hence

$$\begin{aligned} & - \int_{B_R^*} A(|x|) |\psi \varphi'|^p x^P dx - p \int_{B_R^*} A(|x|) |\psi \varphi'|^{p-2} \psi \varphi' \varphi(|x|) \mathcal{R} \psi x^P dx \\ & = - \int_{B_R^*} B(|x|) |u|^p x^P dx \end{aligned}$$

In other words

$$\begin{aligned} & \int_{B_R^*} A(|x|) |\nabla u|^p x^P dx - \int_{B_R^*} B(|x|) |u|^p x^P dx \\ & = \int_{B_R^*} A(|x|) C_p \left(\nabla u, \varphi \nabla \left(\frac{u}{\varphi} \right) \right) x^P dx. \end{aligned}$$

□

4 Hardy–Sobolev Inequality with Monomial Weights—Proof of Theorem 1.8

We first recall the Bliss lemma [7]:

Lemma 4.1 *Let m, n be constants, such that $n > m > 1$ and let $f(x)$ be a real single-valued non-negative function in the interval $0 \leq x < \infty$ such that the integral*

$$J = \int_0^\infty f^m dx$$

taken in the sense of Lebesgue, is finite. Then the integral

$$F = \int_0^x f dt$$

is finite for every x and

$$I = \int_0^\infty \frac{F^n}{x^{n-r}} dx \leq \tau J^{\frac{n}{m}}$$

where $r = \frac{n}{m} - 1$, $\tau = \frac{1}{n-r-1} \left[\frac{r\Gamma(\frac{n}{r})}{\Gamma(\frac{1}{r})\Gamma(\frac{n-1}{r})} \right]^r$. The equality happens for every function of the form $f = c(dx^r + 1)^{-\frac{r+1}{r}}$, and for every function differing from one of these only on a set of values of x of measure zero, but for no others.

Proof of Theorem 1.8 Let

$$HS_{p,s,N,P} = \inf \left\{ \frac{\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx}{\left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^*(s)}}{|x|^s} x^P dx \right)^{\frac{D-P}{D-s}}}; u \in W_0^{1,p}(\mathbb{R}_*^N, x^P dx) \right\}.$$

By Lemma 1.2 and the Hardy–Littlewood inequality, we can assume that u is radial. In this case,

$$\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx = \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right) \int_0^\infty |u_r|^p r^{D-1} dr$$

and

$$\int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^*(s)}}{|x|^s} x^P dx = \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right) \int_0^\infty |u(r)|^{p_D^*(s)} r^{D-1-s} dr.$$

Let $H = \int_0^\infty |u(r)|^{p_D^*(s)} r^{D-1-s} dr$ and $K = \int_0^\infty |u_r|^p r^{D-1} dr$. Let $x = r^{\frac{p-D}{p-1}}$ (i.e. $r = x^{\frac{p-1}{p-D}}$) and $v(x) = u\left(x^{\frac{p-1}{p-D}}\right)$. Then

$$\begin{aligned} dx &= \frac{p-D}{p-1} r^{\frac{p-D}{p-1}-1} dr \\ dr &= \frac{p-1}{p-D} x^{\frac{p-1}{p-D}-1} dx \\ u_r &= v_x \frac{dx}{dr} \\ &= \left(\frac{p-D}{p-1} \right) v_x r^{\left(\frac{p-D}{p-1}\right)-1} \\ &= \left(\frac{p-D}{p-1} \right) v_x x^{\left(\frac{p-1}{p-D}\right)\left(\left(\frac{p-D}{p-1}\right)-1\right)} \\ &= \left(\frac{p-D}{p-1} \right) v_x x^{1-\frac{p-1}{p-D}}. \end{aligned}$$

Therefore

$$H = \int_0^\infty |u(r)|^{p_D^*(s)} r^{D-1-s} dr$$

$$\begin{aligned}
&= - \int_0^\infty |v(x)|^{p_D^*(s)} x^{\left(\frac{p-1}{p-D}\right)(D-1-s)} \frac{p-1}{p-D} x^{\frac{p-1}{p-D}-1} dx \\
&= \frac{p-1}{D-p} \int_0^\infty |v(x)|^{p_D^*(s)} x^{\left(\frac{p-1}{p-D}\right)(D-s)-1} dx
\end{aligned}$$

and

$$\begin{aligned}
K &= \int_0^\infty |u_r|^p r^{D-1} dr \\
&= - \int_0^\infty \left| \left(\frac{p-D}{p-1} \right) v_x x^{1-\frac{p-1}{p-D}} \right|^p x^{\left(\frac{p-1}{p-D}\right)(D-1)} \frac{p-1}{p-D} x^{\frac{p-1}{p-D}-1} dx \\
&= \left(\frac{D-p}{p-1} \right)^{p-1} \int_0^\infty |v_x|^p dx.
\end{aligned}$$

Applying the Bliss lemma for $m = p$, $n = p_D^*(s) = \frac{D-s}{D-p}p$, $r = \frac{p-s}{D-p}$

$$J = \int_0^\infty |v_x|^p dx$$

and

$$\begin{aligned}
I &= \int_0^\infty |v(x)|^{p_D^*(s)} x^{\left(\frac{p-1}{p-D}\right)(D-s)-1} dx = \int_0^\infty |v(x)|^{p_D^*(s)} x^{\frac{p-s}{D-p} - \frac{D-s}{D-p}p} dx, \\
\text{we get } I &\leq \frac{D-p}{(D-s)(p-1)} \left[\frac{\frac{p-s}{D-p} \Gamma\left(\frac{(D-s)p}{p-s}\right)}{\Gamma\left(\frac{D-p}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)} \right]^{\frac{p-s}{D-p}} J^{\frac{D-s}{D-p}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
H &= \frac{p-1}{D-p} I \\
&\leq \frac{p-1}{D-p} \frac{D-p}{(D-s)(p-1)} \left[\frac{\frac{p-s}{D-p} \Gamma\left(\frac{(D-s)p}{p-s}\right)}{\Gamma\left(\frac{D-p}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)} \right]^{\frac{p-s}{D-p}} \left(K \left(\frac{p-1}{D-p} \right)^{p-1} \right)^{\frac{D-s}{D-p}} \\
&= \frac{1}{(D-s)} \left[\frac{\frac{p-s}{D-p} \Gamma\left(\frac{(D-s)p}{p-s}\right)}{\Gamma\left(\frac{D-p}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)} \right]^{\frac{p-s}{D-p}} \left(\frac{p-1}{D-p} \right)^{\left(\frac{D-s}{D-p}\right)(p-1)} K^{\frac{D-s}{D-p}}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
HS_{p,s,N,p} &= \inf \frac{\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx}{\left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^*(s)}}{|x|^s} x^P dx \right)^{\frac{D-p}{D-s}}} \\
&= \inf \frac{\left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right) \int_0^\infty |u_r|^p r^{D-1} dr}{\left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{\frac{D-p}{D-s}} \left(\int_0^\infty |u(r)|^{p_D^*(s)} r^{D-1-s} dr \right)^{\frac{D-p}{D-s}}}
\end{aligned}$$

$$\begin{aligned}
 &= \inf \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{\frac{p-s}{D-s}} \frac{K}{H^{\frac{D-p}{D-s}}} \\
 &= \inf \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{\frac{p-s}{D-s}} \left(\frac{K^{\frac{D-s}{D-p}}}{H} \right)^{\frac{D-p}{D-s}} \\
 &= \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{\frac{p-s}{D-s}} \\
 &\quad \times \left[(D-s) \left[\frac{\Gamma\left(\frac{D-p}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)}{\frac{p-s}{D-p} \Gamma\left(\frac{(D-s)p}{p-s}\right)} \right]^{\frac{p-s}{D-p}} \left(\frac{D-p}{p-1}\right)^{\left(\frac{D-s}{D-p}\right)(p-1)} \right]^{\frac{D-p}{D-s}} \\
 &= \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \right)^{\frac{p-s}{D-s}} (D-s)^{\frac{D-p}{D-s}} \left(\frac{D-p}{p-1}\right)^{p-1} \\
 &\quad \left[\frac{\Gamma\left(\frac{D-p}{p-s}\right) \Gamma\left(\frac{(D-s)p-(D-p)}{p-s}\right)}{\frac{p-s}{D-p} \Gamma\left(\frac{(D-s)p}{p-s}\right)} \right]^{\frac{p-s}{D-s}}.
 \end{aligned}$$

The infimum can be achieved at functions of the form $v(x) = \int_0^x c \left(d\lambda^{\frac{p-s}{D-p}} + 1 \right)^{-\frac{D-s}{p-s}}$
 $d\lambda = c \left(d + x^{-\frac{p-s}{D-p}} \right)^{-\frac{D-p}{p-s}}$, $c \in \mathbb{R}$, $d > 0$. That is $u(r) = v\left(r^{\frac{p-D}{p-1}}\right) =$
 $c \left(d + r^{-\frac{p-D}{p-1} \frac{p-s}{D-p}} \right)^{-\frac{D-p}{p-s}} = c \left(d + r^{\frac{p-s}{p-1}} \right)^{-\frac{D-p}{p-s}}.$

We now compute $\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma$. Recall that in the polar coordinate, we set

$$\begin{cases} x_1 = r \cos \theta_1, \\ x_2 = r \sin \theta_1 \cos \theta_2, \\ x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ \dots \\ x_{N-2} = r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-3} \cos \theta_{N-2}, \\ x_{N-1} = r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-3} \sin \theta_{N-2} \cos \theta_{N-1}, \\ x_N = r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-3} \sin \theta_{N-2} \sin \theta_{N-1}. \end{cases}$$

Here $r \geq 0$, $\theta_i \in [0; \pi]$, $i = 1, 2, \dots, N-2$, and $\theta_{N-1} \in [0; 2\pi]$. Also

$$\frac{\partial(x_1, \dots, x_N)}{\partial(r, \theta_1, \dots, \theta_{N-1})} = r^{N-1} \sin^{N-2} \theta_1 \dots \sin^2 \theta_{N-3} \sin \theta_{N-2}.$$

Now let k be the number of strictly positive entries of P . Then

$$\begin{aligned}
 &\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) \, d\sigma \\
 &= \frac{1}{2^k} \int_{\mathbb{S}^{N-1}} |\varphi_P(\sigma)| \, d\sigma
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^k} \left[\int_0^\pi \left(\prod_{i=1}^{N-2} \left| (\sin \theta_i)^{\sum_{j=i+1}^N P_j} \cos^{P_i} \theta_i \right| \right) \sin^{N-2} \theta_1 \dots \sin^2 \theta_{N-3} \sin \theta_{N-2} d\theta_1 \dots d\theta_{N-2} \right. \\
&\quad \left. \times \int_0^{2\pi} \left| \sin^{P_N} \theta_{N-1} \cos^{P_{N-1}} \theta_{N-1} \right| d\theta_{N-1} \right] \\
&= \frac{1}{2^{k-1}} \prod_{i=1}^{N-1} \int_0^\pi (\sin \theta_i)^{N-i-1+\sum_{j=i+1}^N P_j} \left| \cos^{P_i} \theta_i \right| d\theta_i.
\end{aligned}$$

Note that for $s > -1$ and $t > -1$, we have that for $\lambda = \cos^2 \theta$:

$$\begin{aligned}
\int_0^\pi \sin^s \theta |\cos^t \theta| d\theta &= 2 \int_0^{\frac{\pi}{2}} \sin^s \theta \cos^t \theta d\theta \\
&= \int_0^1 (1-\lambda)^{\frac{s-1}{2}} \lambda^{\frac{t-1}{2}} d\lambda \\
&= B\left(\frac{s+1}{2}, \frac{t+1}{2}\right) \\
&= \frac{\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{s+t+2}{2}\right)}
\end{aligned}$$

where $B(x, y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda$ is the beta function.

Therefore for $i = 1, 2, \dots, N-1$, we have

$$\begin{aligned}
\int_0^\pi (\sin \theta_i)^{N-i-1+\sum_{j=i+1}^N P_j} \left| \cos^{P_i} \theta_i \right| d\theta_i &= \frac{\Gamma\left(\frac{N-i-1+\sum_{j=i+1}^N P_{j+1}}{2}\right) \Gamma\left(\frac{P_i+1}{2}\right)}{\Gamma\left(\frac{N-i-1+\sum_{j=i+1}^N P_j+P_i+2}{2}\right)} \\
&= \frac{\Gamma\left(\frac{D-i-\sum_{j=1}^i P_j}{2}\right) \Gamma\left(\frac{P_i+1}{2}\right)}{\Gamma\left(\frac{D-(i-1)-\sum_{j=1}^{i-1} P_j}{2}\right)}.
\end{aligned}$$

Hence

$$\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma$$

$$\begin{aligned}
 &= \frac{1}{2^{k-1}} \prod_{i=1}^{N-1} \frac{\Gamma\left(\frac{D-i-\sum_{j=1}^i P_j}{2}\right) \Gamma\left(\frac{P_i+1}{2}\right)}{\Gamma\left(\frac{D-(i-1)-\sum_{j=1}^{i-1} P_j}{2}\right)} \\
 &= \frac{1}{2^{k-1}} \frac{\Gamma\left(\frac{P_1+1}{2}\right) \Gamma\left(\frac{P_2+1}{2}\right) \cdots \Gamma\left(\frac{P_n+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)}.
 \end{aligned}$$

□

5 Proofs of Lemma 1.1 and Some Consequences of Our Main Results

Proof of Lemma 1.1 Let $a = |x|$ and $b = |x - y|$, then $C_p(x, y) \geq a^p + (p - 1)b^p - pb^{p-2}ab = Q(a, b)$. If $b = 0$, it is obvious that $Q(a, 0) = a^p \geq 0$ and $Q(a, 0) = 0$ iff $a = 0$. If $b \neq 0$, then by homogeneity, we can assume $b = 1$. In this case, $Q(a, 1) = a^p - pa + (p - 1)$. Noting that $\partial_a Q(a, 1) = pa^{p-1} - p$. Hence $Q(a, 1) \geq Q(1, 1) = 0$. Note that $C_p(x, y) = 0$ iff $|x| = |x - y|$, and $(x - y) \cdot x = |x - y||x|$. Hence, $x - y = x$, i.e., $y = 0$. □

Proof of Corollary 1.2 We have that $\left(r^{D-1}r^{2-D}, r^{D-1}r^{2-D}\frac{z_0^2}{R^2}\right)$ is a 2-Bessel pair on $(0, R)$ with $\varphi(r) = J_0\left(\frac{z_0}{R}r\right)$. Here $z_0 = 2.4048\dots$ is the first zero of the Bessel function $J_0(z)$. See [40, page 6]. By the asymptotic property $J_0(s) \sim 1$ for $s \ll 1$, we have

$$\begin{aligned}
 \int_{B_R^*} \frac{|J_0\left(\frac{z_0}{R}|x|\right)|^2}{|x|^{D-2}} x^P dx &\sim \int_0^R \left|J_0\left(\frac{z_0}{R}r\right)\right|^2 r dr \\
 &\sim \int_0^{z_0} |J_0(s)|^2 s ds \\
 &< \infty.
 \end{aligned}$$

Therefore, $J_0\left(\frac{z_0}{R}|x|\right)$ is the optimizer for the Poincaré inequalities (1.9). □

Proof of Corollary 1.3 We can check that $\left(r^{D-1}r^{-\lambda}, r^{D-1}r^{-\lambda}\left[\left(\frac{(D-\lambda-2)^2}{4} - \alpha^2\right)\frac{1}{r^2} + \frac{z_\alpha^2}{R^2}\right]\right)$ is a 2-Bessel pair on $(0, R)$ with $\varphi(r) = r^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}r\right)$. Indeed, in this case

$$\varphi'(r) = \frac{2-D+\lambda}{2} r^{-\frac{D-\lambda}{2}} J_\alpha\left(\frac{z_\alpha}{R}r\right) + \frac{z_\alpha}{R} r^{\frac{2-D+\lambda}{2}} J'_\alpha\left(\frac{z_\alpha}{R}r\right).$$

Therefore

$$\begin{aligned}
 & \left(r^{D-\lambda-1} \left[\frac{2-D+\lambda}{2} r^{-\frac{D-\lambda}{2}} J_\alpha \left(\frac{z_\alpha}{R} r \right) + \frac{z_\alpha}{R} r^{\frac{2-D+\lambda}{2}} J'_\alpha \left(\frac{z_\alpha}{R} r \right) \right] \right)' \\
 &= \left(\frac{2-D+\lambda}{2} r^{\frac{D-\lambda}{2}-1} J_\alpha \left(\frac{z_\alpha}{R} r \right) + \frac{z_\alpha}{R} r^{\frac{D-\lambda}{2}} J'_\alpha \left(\frac{z_\alpha}{R} r \right) \right)' \\
 &= \frac{z_\alpha^2}{R^2} r^{\frac{D-\lambda}{2}} J''_\alpha \left(\frac{z_\alpha}{R} r \right) + \left(\frac{2-D+\lambda}{2} \frac{z_\alpha}{R} r^{\frac{D-\lambda}{2}-1} + \frac{D-\lambda}{2} \frac{z_\alpha}{R} r^{\frac{D-\lambda}{2}-1} \right) J'_\alpha \left(\frac{z_\alpha}{R} r \right) \\
 &\quad + \left(\frac{2-D+\lambda}{2} \right) \left(\frac{D-\lambda}{2} - 1 \right) r^{\frac{D-\lambda}{2}-2} J_\alpha \left(\frac{z_\alpha}{R} r \right) \\
 &= \frac{z_\alpha^2}{R^2} r^{\frac{D-\lambda}{2}} J''_\alpha \left(\frac{z_\alpha}{R} r \right) + \frac{z_\alpha}{R} r^{\frac{D-\lambda}{2}-1} J'_\alpha \left(\frac{z_\alpha}{R} r \right) - \left(\frac{D-\lambda-2}{2} \right)^2 r^{\frac{D-\lambda}{2}-2} J_\alpha \left(\frac{z_\alpha}{R} r \right) \\
 &= r^{\frac{D-\lambda}{2}-2} \left[\frac{z_\alpha^2}{R^2} r^2 J''_\alpha \left(\frac{z_\alpha}{R} r \right) + \frac{z_\alpha}{R} r J'_\alpha \left(\frac{z_\alpha}{R} r \right) \right] - \left(\frac{D-\lambda-2}{2} \right)^2 r^{\frac{D-\lambda}{2}-2} J_\alpha \left(\frac{z_\alpha}{R} r \right) \\
 &= -r^{\frac{D-\lambda}{2}-2} \left(\frac{z_\alpha^2}{R^2} r^2 - \alpha^2 \right) J_\alpha \left(\frac{z_\alpha}{R} r \right) - \left(\frac{D-\lambda-2}{2} \right)^2 r^{\frac{D-\lambda}{2}-2} J_\alpha \left(\frac{z_\alpha}{R} r \right) \\
 &= -r^{D-\lambda-1} r^{\frac{2-D+\lambda}{2}} J_\alpha \left(\frac{z_\alpha}{R} r \right) \left[\left(\frac{(D-\lambda-2)^2}{4} - \alpha^2 \right) \frac{1}{r^2} + \frac{z_\alpha^2}{R^2} \right].
 \end{aligned}$$

Note that in this case, $J_\alpha(s) \sim \frac{(\frac{s}{2})^\alpha}{\Gamma(\alpha+1)}$ for $s \ll 1$. Therefore for $\alpha > 0$, we have

$$\begin{aligned}
 \int_{B_R^*} \frac{\left| |x|^{\frac{2-D+\lambda}{2}} J_\alpha \left(\frac{z_\alpha}{R} |x| \right) \right|^2}{|x|^\lambda} x^P dx &\sim \int_0^R \left| J_\alpha \left(\frac{z_\alpha}{R} r \right) \right|^2 r dr \\
 &\sim \int_0^{z_\alpha} |J_\alpha(s)|^2 s ds < \infty
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{B_R^*} \frac{\left| |x|^{\frac{2-D}{2}} J_\alpha \left(\frac{z_\alpha}{R} |x| \right) \right|^2}{|x|^{\lambda+2}} x^P dx &\sim \int_0^R \left| J_\alpha \left(\frac{z_\alpha}{R} r \right) \right|^2 r^{-1} dr \\
 &\sim \int_0^{z_\alpha} |J_\alpha(s)|^2 s^{-1} ds < \infty
 \end{aligned}$$

while

$$\begin{aligned}
 \int_{B_R^*} \frac{\left| |x|^{\frac{2-D}{2}} J_0 \left(\frac{z_0}{R} |x| \right) \right|^2}{|x|^{\lambda+2}} x^P dx &\sim \int_0^R \left| J_0 \left(\frac{z_0}{R} r \right) \right|^2 r^{-1} dr \\
 &\sim \int_0^{z_\alpha} |J_0(s)|^2 s^{-1} ds = \infty.
 \end{aligned}$$

Hence, for $0 < \alpha \leq \frac{D-\lambda-2}{2}$, $|x|^{\frac{2-D+\lambda}{2}} J_\alpha\left(\frac{z_0}{R}|x|\right)$ is the optimizer of the Brezis–Vázquez inequalities (1.10). However, when $\alpha = 0$, there is no extremizer for the Brezis–Vázquez inequalities (1.10) though we still can consider $|x|^{\frac{2-D+\lambda}{2}} J_0\left(\frac{z_0}{R}|x|\right)$ as the “virtual” optimizer. $\square$

Proof of Corollary 1.4 With

$$\varphi(r) = R^{-\frac{D-p-\lambda}{p}} \left(\left(\frac{r}{R} \right)^{-\frac{D-p-\lambda}{p-1}} - 1 \right)^{\frac{p-1}{p}} = \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}},$$

we have

$$\varphi'(r) = -\frac{D-p-\lambda}{p} \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{-\frac{1}{p}} r^{\frac{1+\lambda-D}{p-1}}$$

and

$$\begin{aligned} & \left(r^{D-1} \frac{1}{r^\lambda} |\varphi'(r)|^{p-2} \varphi'(r) \right)' \\ &= -\left(\frac{D-p-\lambda}{p} \right)^{p-1} \left[r^{D-1-\lambda} \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{-\frac{p-1}{p}} r^{1+\lambda-D} \right]' \\ &= -\left(\frac{D-p-\lambda}{p} \right)^{p-1} \left[\left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{-\frac{p-1}{p}} \right]' \\ &= -\left(\frac{D-p-\lambda}{p} \right)^p \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{1-2p}{p}} r^{\frac{1+\lambda-D}{p-1}} \\ &= -r^{D-1} \left(\frac{D-p-\lambda}{p} \right)^p \frac{1}{r^{D-1+\frac{D-1-\lambda}{p-1}} \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^p} \\ & \quad \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{(p-1)^2}{p}} \\ &= -r^{D-1} \left(\frac{D-p-\lambda}{p} \right)^p \frac{1}{r^{\lambda+p} \left(1 - \left(\frac{R}{r} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} \left(r^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{(p-1)^2}{p}}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{B_R^*} \frac{1}{|x|^{\lambda+p} \left(1 - \left(\frac{R}{|x|} \right)^{-\frac{D-p-\lambda}{p-1}} \right)^p} \left| \left(|x|^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}} \right)^{\frac{p-1}{p}} \right|^p x^P dx \\ & \sim \int_0^1 \frac{1}{r} \frac{1}{1 - r^{\frac{D-p-\lambda}{p-1}}} dr \end{aligned}$$

which is infinite. Therefore (1.11) has no extremizer but admits $u(x) = \left(|x|^{-\frac{D-p-\lambda}{p-1}} - R^{-\frac{D-p-\lambda}{p-1}}\right)^{\frac{p-1}{p}}$ as the “virtual” optimizer. $\square$

Proof of Corollary 1.5 Indeed, since $\varphi(r) = \left|\ln \frac{r}{R}\right|^{\frac{p-1}{p}}$ and $\varphi'(r) = -\frac{p-1}{p} \left|\ln \frac{r}{R}\right|^{-\frac{1}{p}} \frac{1}{r}$, we have

$$\begin{aligned} & \left(-r^{p-1} \left| -\frac{p-1}{p} \left| \ln \frac{r}{R} \right|^{-\frac{1}{p}} \frac{1}{r} \right|^{p-2} \frac{p-1}{p} \left| \ln \frac{r}{R} \right|^{-\frac{1}{p}} \frac{1}{r} \right)' \\ &= - \left(\frac{p-1}{p} \right)^{p-1} \left(\left| \ln \frac{r}{R} \right|^{-\frac{p-1}{p}} \right)' \\ &= - \left(\frac{p-1}{p} \right)^p \left| \ln \frac{r}{R} \right|^{\frac{1}{p}-2} \frac{1}{r} \\ &= -r^{D-1} \left(\frac{p-1}{p} \right)^p \frac{1}{r^D \left| \ln \frac{r}{R} \right|^p} \left| \ln \frac{r}{R} \right|^{\frac{(p-1)^2}{p}}. \end{aligned}$$

Similarly, (1.12) has no extremizer and has $u(x) = \left|\ln \frac{|x|}{R}\right|^{\frac{p-1}{p}}$ as its “virtual” optimizer. $\square$

Proof of Corollary 1.7 By Lemma 1.2, to prove (1.13), we can assume that u is radial and decreasing function. Let us denote in the polar coordinate that

$$\begin{aligned} x^P &= r^{|P|} \varphi_P(\sigma), \\ |P| &= P_1 + \cdots + P_N, \end{aligned}$$

and

$$\mathbb{S}_*^{N-1} = B_R^* \cap \mathbb{S}^{N-1}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}_*^N} |u|^q x^P dx &\geq \int_{B_R^*} |u|^q x^P dx \\ &= \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \int_0^R |u|^q r^{|P|} r^{N-1} dr \\ &\geq |u(R)|^q \frac{R^D}{D} \int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma. \end{aligned}$$

This implies

$$|u(x)| \leq D^{\frac{1}{q}} |x|^{-\frac{D}{q}} \left(\int_{\mathbb{R}_*^N} |u|^q x^P dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{1}{q}}.$$

In particular, by choosing $q = \frac{Dp}{D-p}$, we get

$$|u(x)| \leq D^{\frac{D-p}{Dp}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{D-p}{Dp}} |x|^{-\frac{D-p}{p}} \left(\int_{\mathbb{R}_*^N} |u|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{D-p}{Dp}}.$$

and

$$|u(x)|^{\frac{p^2}{D-p}} \leq D^{\frac{p}{D}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D}} |x|^{-p} \left(\int_{\mathbb{R}_*^N} |u|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p}{D}}.$$

Equivalently,

$$|u(x)|^{\frac{Dp}{D-p}} x^P \leq D^{\frac{p}{D}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D}} |x|^{-p} \left(\int_{\mathbb{R}_*^N} |u|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p}{D}} |u|^p x^P.$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}_*^N} |u(x)|^{\frac{Dp}{D-p}} x^P dx \\ & \leq D^{\frac{p}{D}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D}} \left(\int_{\mathbb{R}_*^N} |u|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p}{D}} \int_{\mathbb{R}_*^N} \frac{|u|^p}{|x|^p} x^P dx \\ & \leq \left| \frac{p}{D-p} \right|^p D^{\frac{p}{D}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D}} \left(\int_{\mathbb{R}_*^N} |u|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p}{D}} \int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx. \end{aligned}$$

In other words,

$$\left(\int_{\mathbb{R}_*^N} |u(x)|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{D-p}{Dp}} \leq \left| \frac{p}{D-p} \right| D^{\frac{1}{D}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{1}{D}} \left(\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{1}{p}}.$$

As a consequence, we can also derive the Hardy–Sobolev inequality with monomial weight. Indeed, we have

$$\begin{aligned} & \int_{\mathbb{R}_*^N} \frac{|u(x)|^{p_D^*(s)}}{|x|^s} x^P dx \\ & = \int_{\mathbb{R}_*^N} \frac{|u(x)|^s}{|x|^s} \left(x^P \right)^{\frac{s}{p}} |u(x)|^{p_D^*(s)-s} \left(x^P \right)^{\frac{p-s}{p}} dx \\ & \leq \left(\int_{\mathbb{R}_*^N} \left(\frac{|u(x)|^s}{|x|^s} \left(x^P \right)^{\frac{s}{p}} \right)^{\frac{p}{s}} dx \right)^{\frac{s}{p}} \left(\int_{\mathbb{R}_*^N} \left(|u(x)|^{p_D^*(s)-s} \left(x^P \right)^{\frac{p-s}{p}} \right)^{\frac{p}{p-s}} dx \right)^{\frac{p-s}{p}} \\ & = \left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^p}{|x|^p} x^P dx \right)^{\frac{s}{p}} \left(\int_{\mathbb{R}_*^N} |u(x)|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p-s}{p}} \end{aligned}$$

Using the Hardy inequality and Sobolev inequality with monomial weights, we get

$$\begin{aligned}
 & \left(\int_{\mathbb{R}_*^N} \frac{|u(x)|^p}{|x|^p} x^P dx \right)^{\frac{s}{p}} \left(\int_{\mathbb{R}_*^N} |u(x)|^{\frac{Dp}{D-p}} x^P dx \right)^{\frac{p-s}{p}} \\
 & \leq \left(\left| \frac{p}{D-p} \right|^p \int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{s}{p}} \\
 & \quad \times \left(\left| \frac{p}{D-p} \right|^{\frac{Dp}{D-p}} D^{\frac{p}{D-p}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D-p}} \left(\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{1}{p} \frac{Dp}{D-p}} \right)^{\frac{p-s}{p}} \\
 & = \left| \frac{p}{D-p} \right|^s \left(\left| \frac{p}{D-p} \right|^{\frac{Dp}{D-p}} D^{\frac{p}{D-p}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p}{D-p}} \right)^{\frac{p-s}{p}} \\
 & \quad \times \left(\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{s}{p}} \left(\left(\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{D}{D-p}} \right)^{\frac{p-s}{p}} \\
 & = \left| \frac{p}{D-p} \right|^s \left| \frac{p}{D-p} \right|^{\frac{D(p-s)}{D-p}} D^{\frac{p-s}{D-p}} \left(\int_{\mathbb{S}_*^{N-1}} \varphi_P(\sigma) d\sigma \right)^{-\frac{p-s}{D-p}} \left(\int_{\mathbb{R}_*^N} |\nabla u|^p x^P dx \right)^{\frac{p^*_{D(s)}}{p}}.
 \end{aligned}$$

□

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